Gaussian Effective Potential in Light Front ϕ^4_{1+1}

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Abstract

Gaussian effective potential is obtained for ϕ^4_{1+1} quantized on a light front. It coincides with the one obtained previously within the equal time quantization. The computation of the paper substantiates the claim that light front quantization reproduces the phase structure of the theory implied by the equal time quantization.

Gaussian effective potential (GEP) can be computed nonperturbatively for any theory whose Hamiltonian is a polynomial in canonical variables. Its meaning is discussed in [1], where it was demonstrated that, e.g., for ϕ^4_{1+1}, GEP has nontrivial minima at nonzero value of the field that become absolute minima beyond a critical value of the coupling. The critical value of the coupling predicted by GEP for ϕ^4_{1+1} is in agreement with the critical value obtained in the lattice computation [2]. The treatment in [1] is performed within the equal time quantization.

There is an alternative approach to quantization of fields possessing a number of advantages. It is the so-called light front quantization (for a review, see [3]). One of the objections against this scheme of quantization is that it has troubles in reproducing the known facts about phase structure of quantum field theories. A good test case is the theory ϕ^4_{1+1}, because it is proved rigorously for this theory that there is a spontaneous breaking of the reflection symmetry $\phi \to -\phi$. It takes place when the coupling goes beyond a critical value.

It can be demonstrated heuristically that the phase transition persists in ϕ^4_{1+1} under the light front quantization (see [4]). In [4], Chang's reasoning [5] was extended to the light front quantization. This reasoning implies the presence of a phase transition, but does not predict a critical value of the coupling.

In this note, we continue the line of [4], and demonstrate that GEP obtained under the light front quantization for ϕ^4_{1+1} coincides with the one obtained under the equal time quantization.

Let us recall the derivation of GEP for ϕ^4_{1+1} in equal time quantization. The derivation begins with the expression for the Hamiltonian density,

$$\mathcal{H}(x) = \frac{1}{2}\pi^2(x) + \frac{m^2}{2}\phi^2(x) + \frac{g}{4}\phi^4(x). \tag{1}$$

Here $\pi(x)$ and $\phi(x)$ are the canonical variables. Next we decompose π and ϕ:

$$\phi(x) = \phi_0 + \int \frac{dk}{\sqrt{4\pi\omega(k)}}\big[a(k)e^{-ikx} + a^\dagger(k)e^{ikx}\big], \tag{2}$$

$$\pi(x) = \int \frac{dk}{i\sqrt{4\pi}}\sqrt{\omega(k)}\big[a(k)e^{-ikx} - a^\dagger(k)e^{ikx}\big]. \tag{3}$$

Here ϕ_0 is a constant, and $\omega(k)$ is an even function of k. Regardless of the value of ϕ_0 and behavior of $\omega(k)$, canonical commutation relation

between ϕ and π implies the canonical commutator $[a(l), a^\dagger(k)] = \delta(l-k)$.

Next step is to compute the expectation of $\mathcal{H}$ with respect to the vacuum annihilated by $a(k)$:

$$\begin{aligned} < \mathcal{H}(x) > &= \int \frac{dk}{8\pi}\big(\omega(k) + \frac{m^2}{\omega(k)}\big) + 3g\Big[\int \frac{dk}{8\pi\omega(k)}\Big]^2 \\ &+ 3g\phi_0^2 \int \frac{dk}{8\pi\omega(k)} + \frac{m^2}{2}\phi_0^2 + \frac{g}{4}\phi_0^4. \end{aligned} \quad (4)$$

We now seek for $\omega(k)$ that would minimize the above vacuum expectation at fixed ϕ_0. Requiring variation of the expectation with respect to $\omega(k)$ to vanish, we obtain the equation for $\omega(k)$:

$$\omega^2(k) = m^2 + k^2 + 3g\big(\phi_0^2 + \int \frac{dk}{4\pi\omega(k)}\big). \quad (5)$$

GEP is the above expectation of the Hamiltonian density taken at the solution to Eq. (5). It is a function of $\phi_0^2 \equiv R$ (the variable R is introduced for later convenience). We denote this function $V(R)$.

To get rid of an (infinite) constant, let us consider the derivative $\partial V(R)/\partial R$,

$$\frac{\partial V(R)}{\partial R} = \frac{m^2}{2} + 3g\int \frac{dk}{8\pi\omega(k)} + \frac{g}{2}R \quad (6)$$

(to obtain this, one should notice that due to Eq. (5) the dependence of ω on R can be ignored in the derivation of the rhs). The value of this derivative at $R = 0$ equals by definition half of the renormalized mass squared:

$$m_r^2 = m^2 + 3g\int \frac{dk}{4\pi\bar{\omega}(k)}, \quad (7)$$

where $\bar{\omega}$ is ω at $R = 0$.

We now express Eqs. (5) and (6) in terms of m_r (the aim is to get rid of the ultraviolet divergences):

$$\begin{aligned} \omega^2(k) &= \mu^2(R) + k^2, & (8)\\ \frac{\partial V(R)}{\partial R} &= \frac{\mu^2(R)}{2} - gR, & (9)\\ \mu^2(R) &\equiv m_r^2 + 3g\Big(\int \frac{dk}{4\pi}\Big[\frac{1}{\omega(k)} - \frac{1}{\bar{\omega}(k)}\Big] + R\Big). & (10) \end{aligned}$$

In the last line we introduced a "mass" $\mu(R)$. It is a function of R, and coincides with m_r at $R = 0$. Performing the integration in k explicitly in the definition of $\mu(R)$ (we can do it because of the simple dependence of ω and $\bar{\omega}$ on k), we obtain the equation for $\mu(R)$:

$$\mu^2(R) = 1 - \frac{3g}{4\pi} \log \mu^2(R) + 3gR. \tag{11}$$

Here and from now on we measure all the dimensionfull quantities in the units where $m_r = 1$. The last equation implies that $\mu^2(R)$ is a growing function of positive R; its growth starts from the value $\mu^2(R = 0) = 1$.

Finally, we can integrate the derivative of V in R to obtain the explicit expression for $V(R)$:

$$V(R) = -\frac{gR^2}{2} + \frac{\mu^2(R) - 1}{6g} \Big[\frac{\mu^2(R) - 1}{2} + 1 + \frac{6g}{8\pi}\Big]. \tag{12}$$

The last two equations determine $V(R)$ unambiguously in accord with [1]. For properties of this $V(R)$, see [1].

Let us now repeat the above derivation for the light front quantization. Specifically, we use the scheme suggested in [4]. In this paper, a regularization was introduced in the Lagrangian of the theory, and Hamiltonian quantization was applied to the regularized theory with the initial conditions set at a fixed value of the light front time $x^+ = (x^0 + x^1)/\sqrt{2}$. The regularization involves two parameters, the dimensionless parameter ϵ, and the mass parameter M. The regularization is removed when ϵ vanishes and M goes to infinity. The resulting Hamiltonian density is

$$\mathcal{H}_{lf}(x) = \frac{1}{2}p^2(x) + \frac{\epsilon}{2}\phi_-^2(x) + \frac{m^2}{2}\phi^2(x) + \frac{g}{4}\phi^2(x). \tag{13}$$

Here $p(x) \equiv (\pi(x) - \phi(x))/\sqrt{\epsilon - 4\partial^2/M^2}$, and $\phi_- \equiv \partial\phi$. The derivative is in $x^- \equiv (x^0 - x^1)/\sqrt{2}$. As before, π and ϕ are the canonical variables. The variable x is now x^- (for the equal time quantization, x was identical to x^1).

This Hamiltonian is quite different from the one appearing in equal time quantization (see Eq. (1)). For example, the kinetic term involving $p^2(x)$ is formally divergent when the regularization is removed at fixed canonical variables. We find it to be a remarkable fact that GEP implied by this Hamiltonian coincides with the standard one of Eq. (12) after the regularization is removed.

Repeating literally the above derivation of GEP starting from Eq. (13) for the Hamiltonian, one observes that the only modification of Eqs. (8)–(10) implied by switching over from the Hamiltonian (1) to the Hamiltonian (13) takes place in the equation expressing ω in terms of momentum and mass. For the light front Hamiltonian the expression is as follows:

$$\omega_{lf}^2(k) = k^2 + (\epsilon k^2 + \mu^2(R))(\epsilon + \frac{4k^2}{M^2}). \tag{14}$$

At first glance, the difference in the dependence of ω_{lf} on k with respect to the one taking place for the equal time quantization may cause a difference in the equation for $\mu(R)$, and, consequently, may change the equal time expression for GEP. But this is not the case. To see it, switch to the variable $k_{lf} = k/\sqrt{\epsilon}$ in the integrals over k involved in Eq. (10), and neglect all the terms formally disappearing in the limit $\epsilon \to 0$, $M \to \infty$. After this, the integral in k_{lf} involved in Eq. (10) becomes identical to the one appearing in equal time quantization. Therefore, the final expression for GEP (Eq. (12)) is reproduced in light front quantization.

We conclude with the following observations. First, under the light front quantization GEP is formed by the modes of low momentum. Specifically, the characteristic scale of the momenta in the integrals $\int dk/\omega_{lf}(k)$ is of the order $\sqrt{\epsilon}$, and, in the limit of the regularization removed, the characteristic momenta vanish. Second, negative momentum modes are as important for forming GEP as the modes with positive momentum. As discussed in [4], the modes with negative momenta correspond to tachyons under the light front quantization. Thus, one cannot ignore tachyons in the computation of GEP. This is in contrast to perturbative computations, where all the integrations in momenta can be restricted to positive momenta, and, therefore, tachyons can be ignored.

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